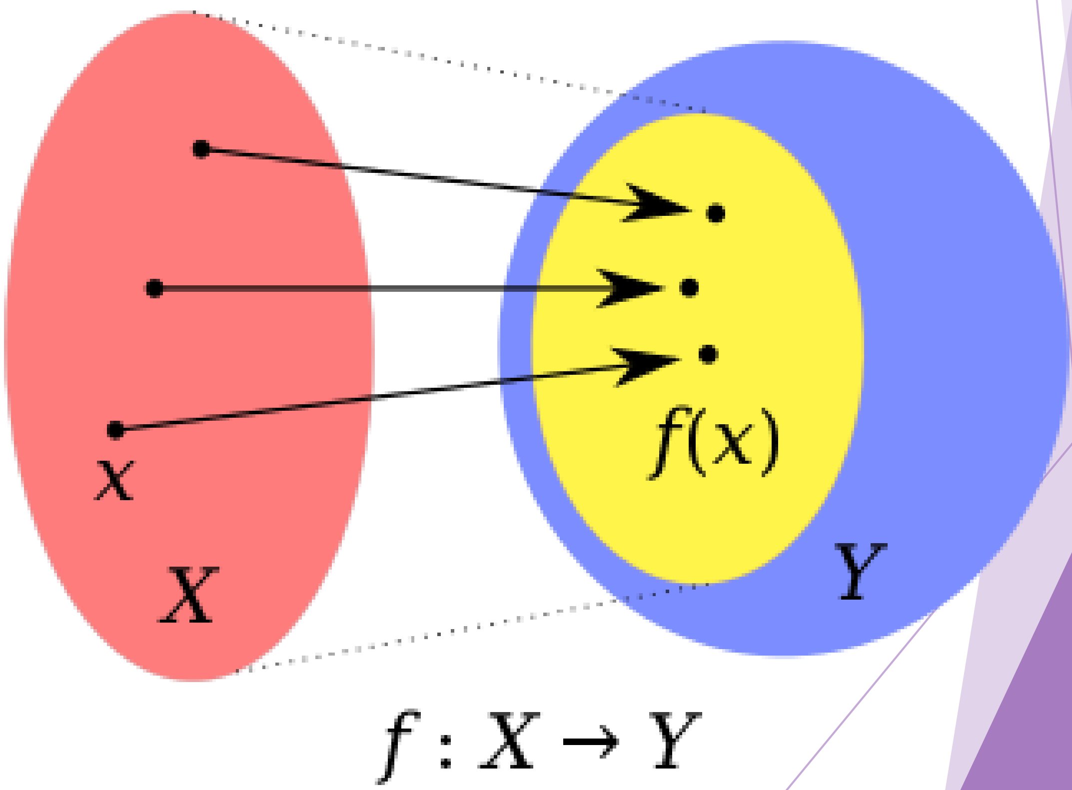
R.B Maths Academy

Question-Answer series

Relation & Function

65 solved problems + 10 practice problems



Contents

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Dear students, I hope you will find the solved problems helpful while preparing for the **Relation** & **Functions** chapter. Please let me know if you find any error in the material. Suggestions are always welcome.

Relations

1. (a) Show that the relation S in the set $A = \{x \in \mathbf{Z} : 0 \leq x \leq 12\}$ given by $S = \{(a, b) : a, b \in A, |a - b| \text{ is divisible by } 4\}$.
is an equivalent relation. Find the set of elements related to 1.

(b) Show that the relation R in the set $A = \{x \in \mathbf{W} : 0 \leq x \leq 17\}$ given by :

i. $R = \{(a, b) : |a - b| \text{ is multiple of } 5\}$

ii. $R = \{(a, b) : a = b\}$

are equivalence relations. Find the set of all elements related to 2 in each case.

(c) If $A = \{0, 1, 2...9\}$ and the relation R on A is defined by $R = \{(x, y) : x, y \in A, y = 2x + 1\}$,
then determine whether the relation R is:
(a) reflexive (b) symmetric (c) transitive

(d) Is the relation R on the set $\mathbf{R}$ of real numbers defined by $R = \{(a, b) : a, b \in \mathbf{R}, 1 + ab \geq 0\}$ transitive? Justify you answer.

(e) Is the relation R on the set $\mathbf{Q}$ of rational numbers defined by $R = \{(x, y) : x, y \in \mathbf{Q}, x < y^2\}$ symmetric? Justify your answer.

Answers

(a) $A = \{x \in \mathbf{Z} : 0 \leq x \leq 12\}$
$S = \{(a, b) : a, b \in A, |a - b| \text{ is divisible by } 4\}$
$|a - b|$ is divisible by 4
$\Rightarrow (a - b) = \pm 4k$ for some $k \in \mathbf{Z}$.

i. $a - a = 0$
Hence, $(a, a) \in S$ and S is reflexive.

ii. Let $(a, b) \in S$
$\Rightarrow (a - b) = \pm 4k$
$\Rightarrow (b - a) = \mp 4k$
$\Rightarrow (b, a) \in S$
Hence, S is symmetric.

iii. Let (a, b) and $(b, c) \in S$
$\Rightarrow |a - b| = 4k_1$ and $|b - c| = 4k_2$ for some $k_1, k_2 \in \mathbf{Z}$
Now, $a - c = a - b + b - c$
$\Rightarrow a - c = (a - b) + (b - c)$
$\Rightarrow a - c = \pm 4k_1 \pm 4k_2 = \pm 4(k_1 + k_2)$
$\Rightarrow |a - c| = 4(k_1 + k_2) \Rightarrow |a - c|$ is divisible by 4,
$\Rightarrow (a, c) \in S$. Hence, S is transitive.

$\therefore$ S is reflexive, symmetric and transitive. Hence, S is an equivalence relation.
Set of elements related to 1 i.e.,
$\{(1, b) : b \in \text{A and } |b - 1| \text{ is divisible by } 4\}$
i.e. $b - 1 = 4k$
$\Rightarrow b = 4k + 1$ where $k \in \mathbf{Z}$ such that $0 \leq b \leq 12$
$\therefore \{5, 9\}$ is the set of elements related to 1.

(b) $A = \{x \in \mathbf{W}, 0 \leq x \leq 17\}$

i. $R = \{(a, b) : |a - b| \text{ is a multiple of } 5\}$

A. $(a, a) = |a - a| = 0 \in R$
So R is reflexive.

B. Let $(a, b) \in R$,
then, $|a - b|$ is a multiple of 5
$\Rightarrow |b - a|$ is a multiple of 5, since $|a - b| = |b - a|$
$\therefore R$ is symmetric.

C. Let $(a, b) \in R$ and $(b, c) \in R$
then, $|a - b| = 5k_1$, $|b - c| = 5k_2$
$\Rightarrow |a - c| = 5k_1 + 5k_2 = 5(k_1 + k_2)$
Hence $(a, c) \in R$.
This implies R is transitive.

So, R is an equivalence relation.
Set of all elements related to 2
i.e. $\{(2, b) : b \in A, |2 - b| \text{ is a multiple of } 5\}$
i.e. $2 - b = 5k \Rightarrow b = 2 - 5k$, where $k \in \mathbf{Z}$ such that $0 \leq b \leq 17$
$\{7, 12, 17\}$ set of elements related to 2.

ii. $R = \{(a, b) : a = b\}$

A. $(a, a) \in R$ since $a = a$
$\therefore R$ is reflexive.

B. Let $(a, b) \in R$
$\Rightarrow a = b$
$\Rightarrow b = a \Rightarrow (b, a) \in R$
Hence R is symmetric.

C. Let (a, b) & $(b, c) \in R$
$\Rightarrow a = b$ and $b = c$
$\Rightarrow a = b = c$
$\Rightarrow a = c$
Hence $(a, c) \in R$.
Thus R is transitive.

$\therefore R$ is reflexive, symmetric and transitive.
Hence, R is an equivalence relation.
Also, $(2, 2) \in R$ as $2 = 2$.
Hence, $\{2\}$ is the set of element related to 2.

(c) $A = \{0, 1, 2...9\}$
$R = \{(x, y) : x, y \in A, y = 2x + 1\}$

i. If $(x, x) \in R$,
then, $x = 2x + 1$
$\Rightarrow -x = 1 \Rightarrow x = -1$
but $-1 \notin A$ So R is not reflexive.

ii. Let $(a, b) \in R$,
$\Rightarrow b = 2a + 1$
$\Rightarrow b - 1 = 2a$
$\Rightarrow \dfrac{b - 1}{2} = a$

So, $(b,a) \notin R$ $[\because (b,a) \in R \Rightarrow a = 2b+1]$
Hence R is not symmetric.

iii. Let (a,b) and $(b,c) \in R$
$\Rightarrow b = 2a+1$ and $c = 2b+1$
To show $(a,c) \in R$, we have to show that $c = 2a+1$.
Now, $c = 2b+1$
$\Rightarrow c = 2(2a+1)+1 = 4a+3 \neq 2a+1$
Hence $(a,b),(b,c) \in R$ does not imply $(a,c) \in R$
Hence R is not transitive.

(d) $R = \{(a,b) : a,b \in \mathbf{R}, 1+ab \geq 0\}$
Let (a,b) and $(b,c) \in R$
$\Rightarrow 1+ab \geq 0$ and $1+bc \geq 0$
R will be transitive if $(a,c) \in R$
i.e. $1+ac \geq 0$
Let us consider the counter example,
$(3,1) \in R$ since $1+(3\times 1) = 4 \geq 0$
and $\left(1, \frac{-1}{2}\right) \in R$ since $1+1\left(-\frac{1}{2}\right) = \frac{1}{2} \geq 0$
But, $\left(3, -\frac{1}{2}\right) \notin R$ since, $1+3\left(-\frac{1}{2}\right) = 1-\frac{3}{2} = -\frac{1}{2} \ngeq 0$
Hence, R is not transitive.

(e) $R = \{(x,y) : x,y \in \mathbf{Q}, x < y^2\}$
To show R is symmetric we have to show that if $(a,b) \in R$ then $(b,a) \in R$
Let us consider the counter example,
$(1,2) \in R$ since, $1 < 2^2$
But $(2,1) \notin R$ since, $2 \nless 1^2$
Hence, R is not symmetric.

2. (a) Show that the relation R in set of real numbers defined as $R = \{(a,b) : a^2+b^2 = 1\}$ is symmetric but neither reflexive nor transitive.

(b) Show that the relation R in set of real numbers defined as $R = \{(a,b) : a \leq b\}$ is reflexive, transitive but not symmetric.

(c) Show that the relation R in set of real numbers defined as $R = \{(a,b) : a \leq b^3\}$ is not an equivalence relation.

(d) Let R be the relation defined on the set $\mathbf{I}$ of all integers by $(x,y) \in R \Leftrightarrow (x-y)$ is divisible by number n. Show that R is an equivalence relation on $\mathbf{I}$.

(e) If a relation R on $\mathbf{Z}$ is defined by $R = \{(a,b) : |a-b| \leq 3\}$ then show that R is reflexive and symmetric but not transitive.

Answers

(a) $R = \{(a,b) : a^2+b^2 = 1\}$

i. Let $(x,x) \in R\ \forall\ x \in \mathbf{R}$
then, $x^2+x^2 = 1$
$\Rightarrow 2x^2 = 1 \Rightarrow x^2 = \frac{1}{2}$
$\Rightarrow x = \pm\frac{1}{\sqrt{2}}$

i.e. Reflexive is possible only when $x = \pm\frac{1}{\sqrt{2}}$ and not for all $x \in \mathbf{R}$

Hence, R is not reflexive.

ii. R is symmetric if for $(a,b) \in R \Rightarrow (b,a) \in R$.
Let $(a,b) \in R$
$\Rightarrow a^2 + b^2 = 1$
$\Rightarrow b^2 + a^2 = 1$
$\Rightarrow (b,a) \in R$
Hence, R is symmetric.

iii. Let (a,b) and $(b,c) \in R$
then $a^2 + b^2 = 1$ ......(i) and $b^2 + c^2 = 1$ ......(ii)
Subtracting (ii) from (i):
$a^2 - c^2 = 0 \Rightarrow a = \pm c$
i.e. $a^2 + c^2 = a^2 + a^2 = 2a^2 \neq 1$ for all $x \in \mathbf{R}$
Hence, $(a,c) \notin R$
So, R is symmetric but neither reflexive nor transitive.

(b) $R = \{(a,b) : a \leq b\}$

i. $x \leq x \;\forall\; x \in R$
Hence, $(x,x) \in R \;\forall\; x \in R$.
So, R is reflexive.

ii. Let $(x,y) \in R$ then,
$x \leq y \not\Rightarrow y \leq x$ for all $x, y \in \mathbf{R}$
$\Rightarrow (y,x) \notin R$
So, R is not symmetric.

iii. Let $(x,y),(y,z) \in R \;\forall\; x,y,z \in \mathbf{R}$.
then, $x \leq y$ and $y \leq z$
$\Rightarrow x \leq y \leq z$
i.e., $x \leq z$
$\Rightarrow (x,z) \in R$ and hence, R is transitive.

So, R is reflexive, transitive but not symmetric.

(c) $R = \{(a,b) : a \leq b^3\}$

i. $(a,a) \notin R$ since, $a \not\leq a^3$ for all $a \in \mathbf{R}$.
For example if $a = -2 \in \mathbf{R}$, then $a > a^3$.
Hence, R is not reflexive.

ii. Let $(a,b) \in R$
then $a \leq b^3 \not\Rightarrow b \leq a^3$ for all $a, b \in \mathbf{R}$
For example, $(1,2) \in R$, as $1 \leq 2^3$, but $(2,1) \notin R$, as $2 > 1^3$.
Hence, R is not symmetric.

iii. Let $(a,b),(b,c) \in R$
then $a \leq b^3$, and $b \leq c^3$
$\Rightarrow a \leq \left(c^3\right)^3 = c^9 \not\Rightarrow a \leq c^3$ for all $a, b, c \in \mathbf{R}$
For example, $\left(3, \frac{3}{2}\right), \left(\frac{3}{2}, \frac{4}{3}\right) \in R$, as $3 \leq \left(\frac{3}{2}\right)^3$ and $\frac{3}{2} \leq \left(\frac{4}{3}\right)^3$. But, $\left(3, \frac{4}{3}\right) \notin R$, as $3 > \left(\frac{4}{3}\right)^3$. Hence, R is not transitive.

Thus, R is not an equivalence relation.

(d) $R \Rightarrow$ relation defined on the set $\mathbf{I}$ of all integers by $(x, y) \in R \Leftrightarrow (x, y)$ is divisible by number n.

i. $x - x = 0$ is divisible by 5.
$\Rightarrow R$ is reflexive.

ii. $(x, y) \in R$
$\Rightarrow (x - y)$ is divisible by n.
$\Rightarrow (y - x)$ is divisible by n.
$\Rightarrow (y, x) \in R$
Hence, R is symmetric.

iii. Let $(x, y) \in R$ and $(y, z) \in R$
$\Rightarrow (x - y)$ is divisible by n and $(y - x)$ is divisible by n.
Now, $(x - y) + (y - z) = x - z$
So, $(x - z)$ is also divisible by n.
$\Rightarrow (x, z) \in R$.
Hence R is transitive.

Thus, R is an equivalence relation.

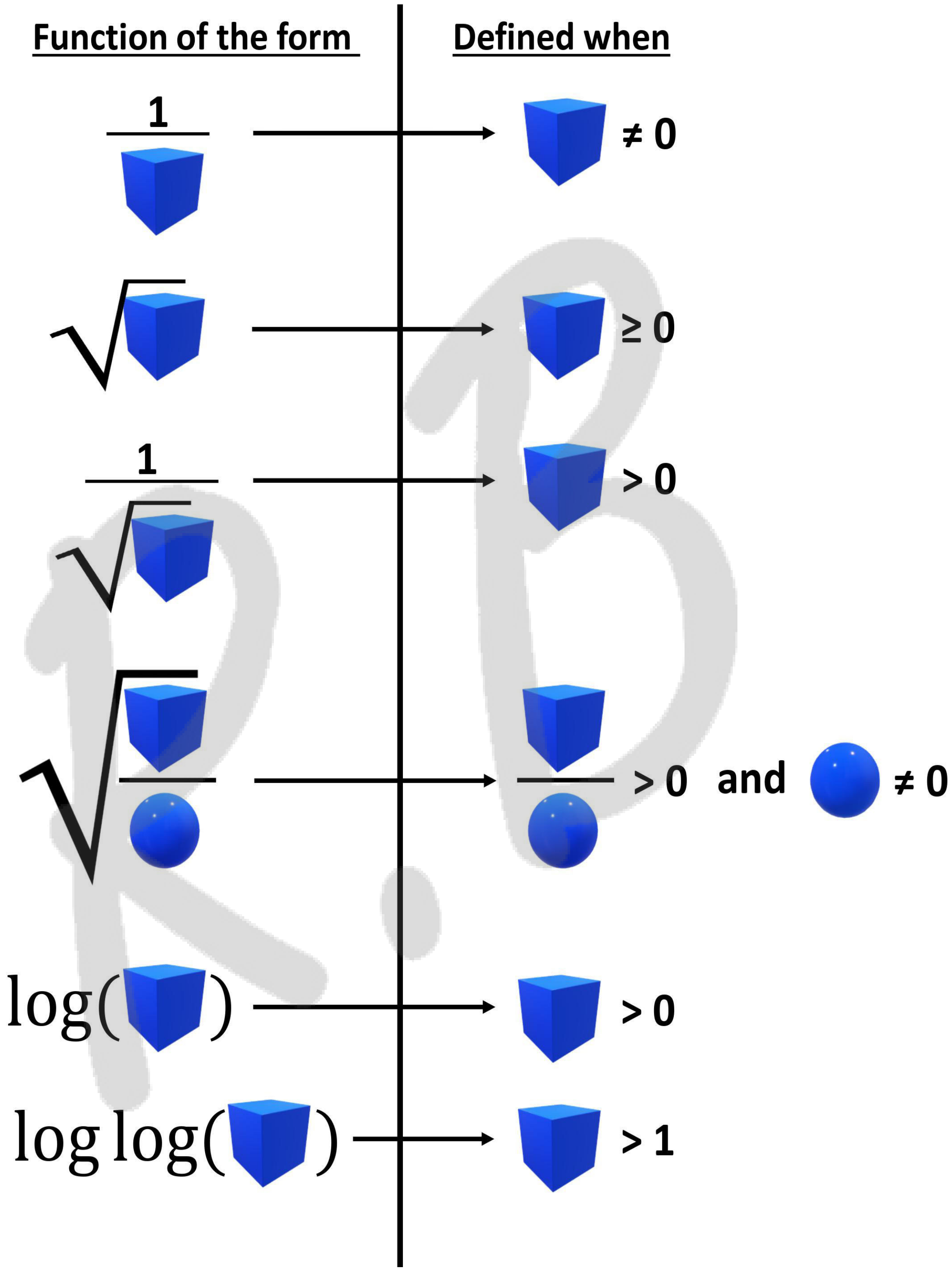
(e) $R = \{(a, b) : |a - b| \leq 3\}$

i. $|a - a| = 0 \leq 3$
$\Rightarrow (a, a) \in R$
So, R is reflexive.

ii. Let $(a, b) \in R$
$\Rightarrow |a - b| \leq 3$
$\Rightarrow |b - a| \leq 3$
$\Rightarrow (b, a) \in R$
So, R is symmetric.

iii. Let (a, b) & $(b, c) \in R$
$\Rightarrow |a - b| \leq 3$ and $|b - c| \leq 3$
Now, $|a - c| = |(a - b) + (b - c)| \leq |a - b| + |b - c| \leq 3 + 3 = 6$
So, $(a, c) \notin R$
Hence, R is not transitive.

Domain and Range



1. Find the domain and range of the following functions:

(a) $f(x) = \dfrac{x^2 - 9}{x - 3}$

(b) $f(x) = \dfrac{x + 1}{2x + 1}$

(c) $f(x) = \sqrt{16 - x^2}$

(d) $f(x) = \dfrac{1}{\sqrt{4 - x^2}}$

(e) $f(x) = |x - 3|$

(f) $f(x) = 3 - |x - 2|$

Answers

(a) $f(x) = \dfrac{x^2 - 9}{x - 3}$

Domain of $f = \mathbf{R} - \{3\}$

Let $y = f(x) = \dfrac{x^2 - 9}{x - 3} \Rightarrow y = x + 3 \quad [\because x \neq 3]$

$\Rightarrow x = y - 3 \in \mathbf{R} - \{3\}$ only if $y \neq 6$.

$\therefore$ Range of $f = \mathbf{R} - \{6\}$.

(b) $f(x) = \dfrac{x + 1}{2x + 1}$

Now, $2x + 1 \neq 0 \implies x \neq -\dfrac{1}{2}$

Hence, Domain of $f = \mathbf{R} - \left\{-\dfrac{1}{2}\right\} = \left(-\infty, \dfrac{-1}{2}\right) \cup \left(\dfrac{-1}{2}, \infty\right)$

Let $y = f(x) = \dfrac{x + 1}{2x + 1}$

$\implies 2xy + y = x + 1$

$\implies 2xy - x = 1 - y$

$\implies x(2y - 1) = 1 - y$

$\implies x = \dfrac{1 - y}{2y - 1}$ Now, $2y - 1 \neq 0 \implies y \neq \dfrac{-1}{2}$

Hence, $y \in \mathbf{R} - \left\{\dfrac{1}{2}\right\}$

$\therefore$ Range of $f = \mathbf{R} - \left\{\dfrac{1}{2}\right\} = \left(-\infty, \dfrac{1}{2}\right) \cup \left(\dfrac{1}{2}, \infty\right)$

(c) $f(x) = \sqrt{16 - x^2}$

f is defined when $16 - x^2 \geq 0$

$\implies (4 - x)(4 + x) \leq 0$

$\implies x \in [-4, 4]$

$\therefore$ Domain of $f = [-4, 4]$

Let $y = \sqrt{16 - x^2}$

Squaring both sides

$y^2 = 16 - x^2$

$\implies x^2 = 16 - y^2$

$\implies x = \pm\sqrt{16 - y^2}$ which exists when $y \in [-4, 4]$

Since, $y = \sqrt{16 - x^2}$ so we must have $y \geq 0$
$\therefore$ Range of $f = [0, 4]$

(d) $f(x) = \dfrac{1}{\sqrt{4 - x^2}}$
f is defined when $4 - x^2 > 0$
$\implies (4 - x)(4 + x) > 0$
$\implies x \in (-2, 2)$
$\therefore$ Domain of $f = (-2, 2)$
Let $y = f(x) = \dfrac{1}{\sqrt{4 - x^2}}$
Squaring both sides,
$y^2 = \dfrac{1}{4 - x^2}$
$\implies 4y^2 - x^2y^2 = 1$
$\implies x^2y^2 = 4y^2 - 1$
$\implies x^2 = \dfrac{4y^2 - 1}{y^2}$
$\implies x = \pm\dfrac{\sqrt{4y^2 - 1}}{y}$ which is defined when $y \neq 0$ and $4y^2 - 1 \geq 0$
i.e., $y \neq 0$ and $\left(y - \dfrac{1}{2}\right)\left(y + \dfrac{1}{2}\right) \geq 0$
i.e., $y \in \left(-\infty, -\dfrac{1}{2}\right] \cup \left[\dfrac{1}{2}, \infty\right)$
But, $y = \dfrac{1}{\sqrt{4 - x^2}} \implies y > 0.$
$\therefore$ Range of $f = \left[\dfrac{1}{2}, \infty\right)$

(e) $f(x) = |x - 3|$
$\therefore$ Domain of $f = \mathbf{R} = (-\infty, \infty)$
Let $y = |x - 3|$
then $y \geq 0 \ \forall \ x \in \mathbf{R}$
Hence, Range of $f = [0, \infty)$

(f) $f(x) = 3 - |x - 2|$
$\therefore$ Domain of $f = \mathbf{R} = (-\infty, \infty)$

$$y = 3 - |x - 2| = \begin{cases} 3 - (x - 2) = 5 - x & , x > 2 \\ 3 - 0 = 3 & , x = 2 \\ 3 + (x - 2) = 1 + x & , x < 2 \end{cases}$$

We know, $|x - 2| \geq 0$ for all $x \in \mathbf{R} \implies -|x - 2| \leq 0$
$\implies 3 - |x - 2| \leq 3$
$\therefore$ Range of $f = (-\infty, 3]$

2. (a) If a real function f is defined by $f(x) = \dfrac{|x| - x}{2x}$, then find its domain and range.

(b) Find the domain and range of the function f defined by $f(x) = \dfrac{|x - 4|}{x - 4}$.

(c) Find the domain of $f(x) = \log(4x - 3)$.

(d) Find the domain of $f(x) = \dfrac{1}{\log(9 - x^2)}$.

Answers

(a) $f(x) = \dfrac{|x| - x}{2x}$

The function is defined when, $2x \neq 0 \implies x \neq 0$

So, Domain of $f = (-\infty, 0) \cup (0, \infty) = \mathbf{R} - \{0\}$

Now,

$$|x| = \begin{cases} x, x > 0 \\ -x, x < 0 \end{cases}$$

(excluding the case of $x = 0$ since it does not exist in the domain of f).

$$\implies |x| - x = \begin{cases} x - x = 0 & , x > 0 \\ -x - x = -2x & , x < 0 \end{cases}$$

Hence, $f(x) = \begin{cases} 0, x > 0 \\ -1, x < 0 \end{cases}$

$\therefore$ Range of $f = \{0, -1\}$.

(b) $f(x) = \dfrac{|x - 4|}{x - 4}$

f is defined when $x - 4 \neq 0$

$\implies x \neq 4$

$\therefore$ Domain of $f = \mathbf{R} - \{4\} = (\infty, 4) \cup (4, \infty)$

Now, $|x - 4| = \begin{cases} (x - 4), x > 4 \\ -(x - 4), x < 4 \end{cases}$

(excluding the case of $x = 4$ since it does not exist in the domain of f).

$\therefore \; f(x) = \dfrac{|x - 4|}{x - 4} = \begin{cases} 1, x > 4 \\ -1, x < 4 \end{cases}$

$\therefore$ Range of $f = \{1, -1\}$

(c) $f(x) = \log(4x - 3)$

f is defined when $4x - 3 > 0$

$\implies 4x > 3$

$\implies x > \dfrac{3}{4}$

$\therefore$ Domain of $f = \left(\dfrac{3}{4}, \infty\right)$

Let, $y = f(x) = \log(4x - 3)$

$\implies 4x - 3 = e^y \implies x = \dfrac{e^y + 3}{4}$ which is defined for all $y \in \mathbf{R}$

$\therefore$ Range of $f = (-\infty, \infty)$

(d) $f(x) = \dfrac{1}{\log(9 - x^2)}$

f to be defined when, $\log(9 - x^2) \neq 0$ and $9 - x^2 > 0$

$\implies 9 - x^2 \neq 1$ and $(3 - x)(3 + x) > 0$

$\implies x \neq \pm 2\sqrt{2}$ and $x \in (-3, 3)$

$\therefore$ Domain of $f = (-3, -2\sqrt{2}) \cup (-2\sqrt{2}, 2\sqrt{2}) \cup (2\sqrt{2}, 3)$

Let, $y = f(x) = \dfrac{1}{\log(9 - x^2)} \implies \log(9 - x^2) = \dfrac{1}{y}$

$\implies 9 - x^2 = e^{\frac{1}{y}}$

$\implies x^2 = 9 - e^{\frac{1}{y}}$ which is defined for all $y \in \mathbf{R} - \{0\}$

$\therefore$ Range of $f = \mathbf{R} - \{0\}$.

3. (a) Find Domain of $f(x) = \dfrac{1}{3 - 2\sin x}$

(b) Find Domain of $f(x) = \dfrac{1}{\sqrt{1 - \cos x}}$

(c) Find Range of $f(x) = 2 - 3\cos x$

(d) Find Range of $f(x) = 2 + 5\sin 3x$

Answers

(a) $f(x) = \dfrac{1}{3 - 2\sin x}$
f is defined when, $3 - 2\sin x \neq 0$
$\implies 2\sin x \neq 3$
$\implies \sin x \neq \dfrac{3}{2}$
We know, $-1 \leq \sin x \leq 1$ for all $x \in \mathbf{R}$.

So, $\sin x \neq \dfrac{3}{2}$ is always true irrespective of any value of x.
$\therefore$ Domain of $f = (-\infty, \infty)$.

(b) $f(x) = \dfrac{1}{\sqrt{1 - \cos x}}$
f is defined when, $1 - \cos x > 0$
$\implies \cos x < 1$
We know, $-1 \leq \cos x \leq 1 \ \forall\ x \in \mathbf{R}$.
Thus, for $\cos x \neq 1$ we must have, $x \neq 2\pi n$ for $n \in \mathbf{Z}$.
$\therefore$ Domain of $f = \mathbf{R} - \{2\pi n : n \in \mathbf{Z}\}$.

(c) $f(x) = 2 - 3\cos x$
$-1 \leq \cos x \leq 1 \quad \forall\ x \in (-\infty, \infty)$
$\implies -3 \leq 3\cos x \leq 3 \quad \forall\ x \in (-\infty, \infty)$
$\implies 3 \geq -3\cos x \geq -3 \quad \forall\ x \in (-\infty, \infty)$
$\implies 2 - 3 \leq 2 - 3\cos x \leq 2 + 3 \quad \forall\ x \in (-\infty, \infty)$
$\implies -1 \leq 2 - 3\cos x \leq 5 \quad \forall\ x \in (-\infty, \infty)$
Hence, $(2 - 3\cos x) \in [-1, 5]$.
$\therefore$ Range of $f = [-1, 5]$.

(d) $f(x) = 2 + 5\sin 3x$
$-1 \leq \sin 3x \leq 1 \quad \forall\ x \in (-\infty, \infty)$
$\implies 2 - 5 \leq 2 + 5\sin 3x \leq 2 + 5 \quad \forall\ x \in (-\infty, \infty)$
$\implies -3 \leq 2 + 5\sin 3x \leq 7 \quad \forall\ x \in (-\infty, \infty)$
Hence, $(2 + 5\sin 3x) \in [-3, 7]\ \forall\ x \in (-\infty, \infty)$.
$\therefore$ Range of $f = [-3, 7]$.

4. (a) Find the Domain of the function $f(x) = \dfrac{1}{5 - 3\sin x}$.

(b) Find the Range of the function $f(x) = \dfrac{1}{2 - \cos x}$.

Answers

(a) $f(x) = \dfrac{1}{5 - 3\sin x}$
f is defined when, $5 - 3\sin x \neq 0$
$\implies \sin x \neq \dfrac{5}{3}$
We know, $-1 \leq \sin x \leq 1 \ \ \forall\, x \in (-\infty, \infty)$
So, Domain of $f = (-\infty, \infty)$.

(b) $f(x) = \dfrac{1}{2 - \cos x}$
$-1 \leq \cos x \leq 1 \quad \forall \ \in (-\infty, \infty)$
$\implies 1 \geq -\cos x \geq -1 \quad \forall \ \in (-\infty, \infty)$
$\implies 2 - 1 \leq 2 - \cos x \leq 2 + 1 \quad \forall \ \in (-\infty, \infty)$
$\implies 1 \leq 2 - \cos x \leq 3 \quad \forall \ \in (-\infty, \infty)$
Hence, $(2 - \cos x) \in [1, 3]$.
$\therefore$ Range f is $[1, 3]$.

5. (a) Find the Domain of the function $f(x) = \sqrt{x-1} + \sqrt{6-x}$.

(b) Find the Domain of definition of $f(x) = \dfrac{1}{\sqrt{|x| - x}}$.

Answers

(a) $f(x) = \sqrt{x-1} + \sqrt{6-x}$
$f(x)$ is defined when, $x - 1 \geq 0$ and $6 - x \geq 0$
$\implies x \geq 1$ and $x \leq 6$ i.e., $1 \leq x \leq 6$
$\therefore$ Domain of $f = [1, 6]$.

(b) $f(x) = \dfrac{1}{\sqrt{|x| - x}}$
$f(x)$ is defined when, $|x| - x > 0$.
$\implies |x| > x$, which is possible only when $x \in (-\infty, 0)$.
Hence, Domain of $f(x)$ is $(-\infty, 0)$.

Types of Functions

1. Show that the function

   (a) $f: \mathbf{N} \to \mathbf{N}$ defined by $f(n) = \begin{cases} n+1, & \text{if n is odd} \\ n-1, & \text{if n is even} \end{cases}$ is both one-one and onto.

   (b) $f: \mathbf{Q} \to \mathbf{Q}$ defined by $f(x) = 3x - 2$ is one-one.

   (c) $f: \mathbf{N} \to \mathbf{N}$ defined by $f(x) = 2x - 1$ is not onto.

   (d) $f: \mathbf{N} \to \mathbf{N}$ defined by $f(m) = m^2 + m + 2$ one-one? Justify your answer.

   (e) $f: \mathbf{Z} \to \mathbf{Z}$ defined by $f(x) = x^2 + x$ is neither one-one nor onto.

Answers

(a) $f(n) = \begin{cases} n+1, \text{if n is odd} \\ n-1, \text{if n is even} \end{cases}$

Case-1: Let $x_1 = 2n_1$ (even), $x_2 = 2n_2 + 1$ (odd) for some $n_1, n_2 \in \mathbf{N}$.
Then $x_1 \neq x_2$
$f(x_1) = 2n_1 - 1$ (odd) and $f(x_2) = 2n_2 + 1 + 1 = 2n_2 + 2 = 2(n+1)$ (even).
Hence, $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$
Case-2: Let $x_1 = 2n_1$ (even) and $x_2 = 2n_2$ (even) for some $n_1, n_2 \in \mathbf{N}$.
Then $x_1 \neq x_2$
$f(x_1) = 2n_1 - 1$ (odd) and $f(x_2) = 2n_2 - 1$ (odd).
Hence, $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$
Case-3: Let $x_1 = 2n_1 + 1$ (odd) and $x_2 = 2n_2 + 1$ (odd) for some $n_1, n_2 \in \mathbf{N}$.
Then $x_1 \neq x_2$
$f(x_1) = 2n_1 + 2$ (even) and $f(x_2) = 2n_2 + 2$ (even).
Hence, $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$
Thus, f is one-one.

Let $y = f(n) = \begin{cases} n+1, \text{if n is odd} \\ n-1, \text{if n is even} \end{cases}$ so, $y \in \mathbf{N}$

Let x_1 be even natural no.
$y = f(x_1) = x_1 - 1$
$\implies x_1 = y + 1$
$\implies y$ must be odd natural no.

and x_2 be odd natural no.
and $y = f(x_2) = x_2 + 1$
$\implies x_2 = y - 1$
$\implies y$ must be even natural no.

So, f is onto.

(b) $f: \mathbf{Q} \to \mathbf{Q}$, $f(x) = 3x - 2$
Let $f(x_1) = f(x_2)$ for $x_1, x_2 \in \mathbf{Q}$
$\implies 3x_1 - 2 = 3x_2 - 2$
$\implies 3x_1 = 3x_2$
$\implies x_1 = x_2$
So, $f(x_1) = f(x_2) \implies x_1 = x_2$
Hence, f is one-one.

(c) $f : \mathbf{N} \to \mathbf{N}, f(x) = 2x - 1$
Now, let $y = 2x - 1$
$\implies y + 1 = 2x \implies x = \dfrac{y+1}{2}$
Now, $y \in \mathbf{N}$ and if y is odd then $x \in \mathbf{N}$
but if y is even then $y + 1$ is odd
and $\dfrac{y+1}{2} \notin \mathbf{N}$ i.e. $x \notin \mathbf{N}$
Hence, f is not onto.

(d) $f : \mathbf{N} \to \mathbf{N}, f(m) = m^2 + m + 2$
Let $f(x_1) = f(x_2)$ for $x_1, x_2 \in \mathbf{N}$
$\implies x_1^2 + x_1 + 2 = x_2^2 + x_2 + 2$
$\implies x_1^2 - x_2^2 + x_1 - x_2 = 0$
$\implies (x_1 - x_2)(x_1 + x_2 + 1) = 0$
Now, $x_1 + x_2 + 1 \neq 0$ since $x_1, x_2 \in \mathbf{N}$
$\therefore (x_1 - x_2) = 0 \implies x_1 = x_2$
So, $f(x_1) = f(x_2) \implies x_1 = x_2$
Hence, f is one-one.

(e) $f : \mathbf{Z} \to \mathbf{Z}, f(x) = x^2 + x$
Let $x_1, x_2 \in \mathbf{Z}$ such that $f(x_1) = f(x_2)$
$\implies x_1^2 + x_1 = x_2^2 + x_2$
$\implies x_1^2 - x_2^2 + x_1 - x_2 = 0$
$\implies (x_1 - x_2)(x_1 + x_2 + 1) = 0$
Either, $x_1 = x_2$ or, $x_1 + x_2 + 1 = 0$
Let us consider the example, $x_1 = 0, x_2 = -1$
$\therefore f(x_1) = 0$ and $f(x_2) = 1 - 1 = 0$
So, for $x_1 \neq x_2$ we are getting $f(x_1) = f(x_2)$
Hence, f is not one-one.
Again let us consider the counter example $y = -1 \in \mathbf{Z}$ (co-domain of f)
Now, $y = f(x) \implies x^2 + x = -1$
As, $x^2 + x + 1 = 0$ has no solution in $\mathbf{Z}$, so there does not exist any $x \in \mathbf{Z}$ (domain of f) such that $f(x) = -1$.
Hence, f is not onto.
Thus, f is neither one-one nor onto.

2. (a) Find whether the following function is surjective, injective or bijective:
$f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^3 - 2, x \in \mathbf{R}$

(b) Show that the function $f : \mathbf{R} - \{3\} \to \mathbf{R} - \{1\}$ defined by $f(x) = \dfrac{x+4}{x-3}$ is a bijective function.

(c) Consider the function $f(x) = x + \dfrac{1}{x}, x \in \mathbf{R}, x \neq 0$ is f one-one?

Answers

(a) $f(x) = x^3 - 2, x \in \mathbf{R}$
Let $x_1, x_2 \in \mathbf{R}$ such that $f(x_1) = f(x_2)$
$\implies x_1^3 - 2 = x_2^3 - 2 \implies x_1^3 - x_2^3 = 0$
$\implies (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = 0$
$\implies$ Either, $x_1 - x_2 = 0$ or, $x_1^2 + x_1x_2 + x_2^2 = 0$ ....(i)

Now, $x_1^2 + x_1x_2 + x_2^2 = x_1^2 + 2x_1\left(\frac{x_2}{2}\right) + \left(\frac{x_2}{2}\right)^2 + \frac{3x_2^2}{4}$

$\implies x_1^2 + x_1x_2 + x_2^2 = \left(x_1 + \frac{x_2}{2}\right)^2 + \frac{3x_2^2}{4} \neq 0$ for all $x_1, x_2 \in \mathbf{R}$

$\therefore$ From (i) we must have, $x_1 = x_2$.

Hence, f is one-one (injective)

Now, let $y = f(x) = x^3 - 2, y \in \mathbf{R}$

$y = x^3 - 2$

$\implies y + 2 = x^3$

$\implies x = \sqrt[3]{y+2} \implies x \in \mathbf{R}$ for all $y \in \mathbf{R}$.

So, f is onto.

$\therefore f$ is both one-one (injective) and onto (surjective).

Hence, f is bijective.

(b) $f : \mathbf{R} - \{3\} \to \mathbf{R} - \{1\}, f(x) = \frac{x+4}{x-3}$

Let $x_1, x_2 \in \mathbf{R} - \{3\}$ such that

$f(x_1) = f(x_2)$

$\implies \frac{x_1+4}{x_1-3} = \frac{x_2+4}{x_2-3}$

$\implies x_1x_2 + 4x_2 - 3x_1 - 12 = x_1x_2 - 3x_2 + 4x_1 - 12$

$\implies 7x_2 = 7x_1$

$\implies x_2 = x_1$

So, f is one-one.

Let $y = f(x) = \frac{x+4}{x-3}$ where $x \in \mathbf{R} - \{3\}$

$\implies xy - 3y = x + 4$

$\implies xy - x = 4 + 3y$

$\implies x(y-1) = 3y + 4$

$\implies x = \frac{3y+4}{y-1}$ as, $y \neq 1$ hence x is well defined

Also, $x = 3$ is not possible as, $3 = \frac{3y+4}{y-1} \implies 3y - 3 = 3y + 4$ (not possible).

So, $x = \frac{3y+4}{y-1} \in \mathbf{R} - \{3\}$

Hence, f is onto.

(c) Let $x_1, x_2 \in \mathbf{R}$ and $f(x_1) = f(x_2)$

$\implies x_1 + \frac{1}{x_1} = x_2 + \frac{1}{x_2} \implies \frac{x_1^2+1}{x_1} = \frac{x_2^2+1}{x_2}$

$\implies x_1^2x_2 + x_2 = x_2^2x_1 + x_1$

$\implies x_1^2x_2 - x_2^2x_1 + (x_2 - x_1) = 0$

$\implies x_1x_2(x_1 - x_2) + (x_2 - x_1) = 0$

$\implies (x_1 - x_2)(x_1x_2 - 1) = 0$

$\implies$ Either, $x_1 - x_2 = 0$ or, $x_1x_2 - 1 = 0$

$\implies x_1 = x_2$ or, $x_2 = \frac{1}{x_1}$

Let us consider the counter example, $x_1 = 2, x_2 = \frac{1}{2}$.

Then $f(x_1) = 2 + \frac{1}{2} = \frac{5}{2}$ and $f(x_2) = \frac{1}{2} + 2 = \frac{5}{2}$.

Hence, f is not one-one.

Composition of Functions

1. Find gof and fog when $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ are defined by:

   (a) $f(x) = 2x - 1$ and $g(x) = x^2 + 3 \ \forall \ x \in \mathbf{R}$

   (b) $f(x) = x + 1$ and $g(x) = |x| \ \forall \ x \in \mathbf{R}$

   (c) $f(x) = \sin(x)$ and $g(x) = 5x^2 \ \forall \ x \in \mathbf{R}$

   (d) $f(x) = x^2 - 3x + 2 \ \forall \ x \in \mathbf{R}$, find fof

Answers

(a) $f(x) = 2x - 1, g(x) = x^2 + 3 \ \forall \ x \in \mathbf{R}$
$gof(x) = g(f(x))$
$= g(2x - 1) \ \forall \ x \in \mathbf{R}$
$= (2x - 1)^2 + 3$
$= 4x^2 - 4x + 1 + 3$
$= 4x^2 - 4x + 4 \ \forall \ x \in \mathbf{R}$
and $fog(x) = f(g(x))$
$= f(x^2 + 3)$
$= 2(x^2 + 3) - 1$
$= 2x^2 + 6 - 1 = 2x^2 + 5 \ \forall \ x \in \mathbf{R}$

(b) $f(x) = x + 1, g(x) = |x| \ \forall \ x \in \mathbf{R}$
$gof(x) = g(f(x)) = g(x + 1) = |x + 1| \ \forall \ x \in \mathbf{R}$
$fog(x) = f(g(x)) = f(|x|) = |x| + 1 \ \forall \ x \in \mathbf{R}$

(c) $f(x) = \sin(x), g(x) = 5x^2 \ \forall \ x \in \mathbf{R}$
$gof(x) = g(f(x))$
$= g(\sin x)$
$= 5(\sin x)^2 = 5\sin^2 x \ \forall \ x \in \mathbf{R}$
$fog(x) = f(g(x)) = f(5x^2) = \sin(5x^2) \ \forall \ x \in \mathbf{R}$

(d) $f(x) = x^2 - 3x + 2 \ \forall \ x \in \mathbf{R}$
$fof(x) = f(f(x))$
$= f(x^2 - 3x + 2)$
$= (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2$
$= x^4 + 9x^2 + 4 - 6x^3 - 12x + 4x^2 - 3x^2 + 9x - 6 + 2$
$= x^4 - 6x^3 + 10x^2 - 3x$

2. (a) $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ are defined by $f(x) = x^2$ and $g(x) = x + 1 \ \forall \ x \in \mathbf{R}$. Show that $gof \neq fog$.

   (b) $f : \mathbf{R} \to \mathbf{R}$ is given by $f(x) = (5 - x^5)^{\frac{1}{5}}$, then find fof.

   (c) If $f(x) = |x|$ and $g(x) = [x] \ \forall \ x \in \mathbf{R}$, find $fog\left(-\frac{5}{3}\right)$.

   (d) $f(x) = x + 7$ and $g(x) = x - 7$, find $fog(7)$.

Answers

(a) $f(x) = x^2, g(x) = x + 1 \ \forall\ x \in \mathbf{R}$
$gof(x) = g(f(x))$
$= g(x^2) = x^2 + 1 \ \forall\ x \in \mathbf{R}$
$fog(x) = f(g(x))$
$= f(x+1) = (x+1)^2 \ \forall\ x \in \mathbf{R}$
$= x^2 + 2x + 1 \ \forall\ x \in \mathbf{R}$
We have, $x^2 + 1 \neq x^2 + 2x + 1$
Hence, $fog \neq gof$

(b) $f(x) = (5 - x^5)^{\frac{1}{5}} \ \forall\ x \in \mathbf{R}$
$fof(x) = f\left((5 - x^5)^{\frac{1}{5}}\right)$
$= \left(5 - \left[(5 - x^5)^{\frac{1}{5}}\right]^5\right)^{\frac{1}{5}}$
$= \left[5 - (5 - x^5)\right]^{\frac{1}{5}}$
$= (x^5)^{\frac{1}{5}} = x \ \forall\ x \in \mathbf{R}$

(c) $f(x) = |x|, g(x) = [x] \ \forall\ x \in \mathbf{R}$
$fog(x) = f(g(x)) = f([x]) = |[x]| \ \forall\ x \in \mathbf{R}$
$fog\left(-\frac{5}{3}\right) = \left|\left[-\frac{5}{3}\right]\right| = |-2| = 2$

(d) $f(x) = x + 7$ and $g(x) = x - 7$
$fog(x) = f(g(x)) = f(x - 7)$
$= x - 7 + 7 = x \ \forall\ x \in \mathbf{R}$
$\therefore fog(7) = 7$

3. (a) $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ are defined by $f(x) = x+1$ and $g(x) = x-1 \ \forall\ x \in \mathbf{R}$. Show that $gof = fog = \mathbf{I}_R$.

(b) Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 7x + 3$. Find function $g : \mathbf{R} \to \mathbf{R}$ such that $gof = fog = \mathbf{I}_R$.

Answers

(a) $f(x) = x + 1, g(x) = x - 1 \ \forall\ x \in \mathbf{R}$
$gof(x) = g(f(x)) = g(x+1) = (x+1) - 1 = x \ \forall\ x \in \mathbf{R}$
$\therefore gof = \mathbf{I}_R$
$fog(x) = f(g(x)) = f(x-1) = x - 1 + 1 = x \ \forall\ x \in \mathbf{R}$
$\therefore fog = \mathbf{I}_R$

(b) $f(x) = 7x + 3$
$gof = \mathbf{I}_R \implies gof(x) = x \implies g(7x+3) = x$
Let, $7x + 3 = y \implies x = \frac{y-3}{7}$
$\therefore g(y) = \frac{y-3}{7}$. Hence, $g(x) = \frac{x-3}{7}$
Also, $fog(x) = f\left(\frac{x-3}{7}\right) = 7.\left(\frac{x-3}{7}\right) + 3 = x - 3 + 3 = x$
$\therefore fog = \mathbf{I}_R$
Hence, $g : \mathbf{R} \to \mathbf{R}$ is defined by $g(x) = \frac{x-3}{7} \ \forall\ x \in \mathbf{R}$ such that $gof = fog = \mathbf{I}_R$

Invertible Functions

1. (a) Show that the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 4x + 5$ is invertible. Also find inverse of f.

   (b) If $A = \mathbf{R} - \{-3\}$ and $B = \mathbf{R} - \{2\}$ and a function $f : A \to B$ be given by $f(x) = \dfrac{2x+1}{x+3}$, then show the function is one-one and onto. Hence find f^{-1}.

   (c) Let $f : \mathbf{N} \to S$ be a funtion defined by $f(x) = 9x^3 + 6x - 5$, where S is the range of f. Show that f is invertible. Find the inverse of f and hence find $f^{-1}(43)$ and $f^{-1}(163)$.

   (d) If function $f : \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = 5x - 3$, then find function $g : \mathbf{R} \to \mathbf{R}$ such that $gof = I_R = fog$.

Answers

(a) $f(x) = 4x + 5$
Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$
$\implies 4x_1 + 5 = 4x_2 + 5 \implies 4x_1 = 4x_2 \implies x_1 = x_2$
$\implies f$ is one-one.
Now, let $y = f(x)$
$\implies y = 4x + 5$
$\implies y - 5 = 4x$
$\implies x = \dfrac{y-5}{4} \in \mathbf{R}$
So, f is onto.
Hence, f is invertible.
Now, $y = f(x)$
$\implies f^{-1}(y) = x$
$\implies f^{-1}(y) = \dfrac{y-5}{4} \quad \left[\because x = \dfrac{y-5}{4}\right]$
$\therefore f^{-1}(x) = \dfrac{x-5}{4}$.

(b) $A = \mathbf{R} - \{-3\}$ and $B = \mathbf{R} - \{2\}$ and $f : A \to B$ is given by $f(x) = \dfrac{2x+1}{x+3}$.
Now, let $f(x_1) = f(x_2)$ for $x_1, x_2 \in A$
$\implies \dfrac{2x_1+1}{x_1+3} = \dfrac{2x_2+1}{x_2+3}$
$\implies 2x_1x_2 + x_2 + 6x_1 + 3 = 2x_1x_2 + x_1 + 6x_2 + 3$
$\implies x_2 + 6x_1 = x_1 + 6x_2$
$\implies 5x_1 = 5x_2$
$\implies x_1 = x_2$
Hence, f is one-one.
Now, let $y = f(x)$
$\implies y = \dfrac{2x+1}{x+3}$
$\implies yx + 3y = 2x + 1$
$\implies yx - 2x = 1 - 3y$
$\implies x(y-2) = 1 - 3y$

$\implies x = \dfrac{1-3y}{y-2}$ which is defined for $y - 2 \neq 0 \implies y \neq 2$
$\implies$ range of $f = \mathbf{R} - \{2\} = B$
$\implies$ range of f = codomain of f.
$\implies f$ is onto.
Thus the function is one-one and onto.
$\implies f$ is invertible.
Now, $f(x) = y$ and f is invertible
$\implies f^{-1}(y) = x$
$\implies f^{-1}(y) = \dfrac{1-3y}{y-2}$
$\implies f^{-1}(x) = \dfrac{1-3x}{x-2}$.

(c) $f : \mathbf{N} \to S, \quad f(x) = 9x^2 + 6x - 5$ where S is the range of f.
Let $f(x_1) = f(x_2)$ for, $x_1, x_2 \in \mathbf{N}$
$\implies 9x_1^2 + 6x_1 - 5 = 9x_2^2 + 6x_2 - 5$
$\implies 9x_1^2 - 9x_2^2 = 6x_2 - 6x_1$
$\implies 9\left(x_1^2 - x_2^2\right) = 6\left(x_2 - x_1\right)$
$\implies 9(x_1 + x_2)(x_1 - x_2) = 6(x_2 - x_1)$
$\implies (x_! - x_2)[9(x_1 + x_2) + 6] = 0$
but, $9(x_1 + x_2) + 6 \neq 0$ since $x_1, x_2 \in \mathbf{N}$
$\implies x_1 - x_2 = 0$
$\implies x_1 = x_2$
So, f is one-one.
Again, let $y = f(x)$
$\implies y = 9x^2 + 6x - 5$
$\implies y = (3x)^2 + 2 \times 3x \times 1 + 1 - 1 - 5$
$\implies y + 6 = (3x + 1)^2$
$\implies 3x + 1 = \sqrt{y+6}$ (negative sign is omitted as $x \in \mathbf{N} \implies x > 0$)
$\implies 3x = \sqrt{y+6} - 1$
$\therefore x = \dfrac{\sqrt{y+6}-1}{3} \in \mathbf{N} \quad \forall\, y \in S$.
Hence, f is onto. $\therefore f$ is invertible.
Now, $y = f(x)$
$\implies f^{-1}(y) = x = \dfrac{\sqrt{y+6}-1}{3}$
$\therefore f^{-1}(43) = \dfrac{\sqrt{43+6}-1}{3} = \dfrac{7-1}{3} = \dfrac{6}{3} = 2$
and $f^{-1}(163) = \dfrac{\sqrt{163+6}-1}{3} = \dfrac{13-1}{3} = \dfrac{12}{3} = 4$

(d) $f : \mathbf{R} \to \mathbf{R},\ f(x) = 5x - 3$
Let $x_1, x_2 \in \mathbf{R}$ such that
$f(x_1) = f(x_2)$
$\implies 5x_1 - 3 = 5x_2 - 3$
$\implies 5x_1 = 5x_2$
$\implies x_1 = x_2$
So, f is one-one.
Let $y = f(x) \implies y = 5x - 3$
$\implies x = \dfrac{y+3}{5} \in \mathbf{R} \quad \forall\, y \in \mathbf{R}$

So, f is onto.
Hence, f is invertible.
Let $y = f(x)$ then,
$f^{-1}(y) = x$
$\implies f^{-1}(y) = \dfrac{y+3}{5}$
$\implies f^{-1}(x) = \dfrac{x+3}{5}$
Then,

$$\begin{aligned} fof^{-1}(x) &= f\left(\frac{x+3}{5}\right) \\ &= 5\left(\frac{x+3}{5}\right) - 3 \\ &= x + 3 - 3 = x \end{aligned}$$

and,

$$\begin{aligned} f^{-1}of(x) &= f^{-1}(5x-3) \\ &= \frac{(5x-3)+3}{5} = x \end{aligned}$$

$\therefore fof^{-1} = \mathbf{I}_R = f^{-1}of$
$\therefore g = f^{-1}$ such that $gof = \mathbf{I}_R = fog$
Hence, $g : \mathbf{R} \to \mathbf{R}$, $\quad g(x) = \dfrac{x+3}{5}$.

2. (a) If $f : \mathbf{R} \to \mathbf{R}$, defined by $f(x) = \dfrac{2x-7}{4}$ is invertible, find f^{-1}.

   (b) If $f : \mathbf{R} - \left\{-\dfrac{4}{3}\right\} \to \mathbf{R} - \left\{\dfrac{4}{3}\right\}$ is a function defined by $f(x) = \dfrac{4x}{3x+4}$, then find f^{-1}.

Answers

(a) $f : \mathbf{R} \to \mathbf{R}$, $f(x) = \dfrac{2x-7}{4}$ is invertible.
Let $y = f(x) = \dfrac{2x-7}{4}$
$\implies \dfrac{4y+7}{2} = x$
Again, let $y = f(x)$
$\implies f^{-1}(y) = x = \dfrac{4y+7}{2}$
Hence, $f^{-1}(x) = \dfrac{4x+7}{2}$

(b) $f(x) = \dfrac{4x}{3x+4}$
Let $x_1, x_2 \in \mathbf{R} - \left\{-\dfrac{4}{3}\right\}$ and $f(x_1) = f(x_2)$
$\implies \dfrac{4x_1}{3x_1+4} = \dfrac{4x_2}{3x_2+4}$
$\implies 12x_1x_2 + 16x_1 = 12x_1x_2 + 16x_2$

$\implies 16x_1 = 16x_2 \implies x_1 = x_2$
So, f is one-one.
Again, let $y = f(x)$
$\implies y = \dfrac{4x}{3x+4}$
$\implies 3xy + 4y = 4x$
$\implies 3xy - 4x = -4y$
$\implies x(3y-4) = -4y$
$\implies x = \dfrac{-4y}{3y-4} = \dfrac{4y}{4-3y}$ which is defined for $y \neq \left\{\dfrac{4}{3}\right\}$
$\implies$ range of $f = \mathbf{R} - \left\{\dfrac{4}{3}\right\}$ = co-domain of f.
Hence, f is onto.
$\therefore f$ is invertible.
Now, $y = f(x) \implies f^{-1}(y) = x = \dfrac{4y}{4-3y}$
$\therefore f^{-1}(x) = \dfrac{4x}{4-3x}$.

3. (a) If $A = \mathbf{R} - \left\{\dfrac{2}{3}\right\}$ and a function $f : A \to A$ is defned by $f(x) = \dfrac{4x+3}{6x-4}$. Show that f is one-one and onto. Hence find f^{-1}.

(b) Consider $f : \mathbf{R}_+ \to [-9, \infty)$ given by $f(x) = 5x^2 + 6x - 9$ where $\mathbf{R}_+$ is the set of all non-negative real numbers. Prove that f is invertible with $f^{-1}(y) = \dfrac{\sqrt{54-5y}-3}{5}$.

Answers

(a) Let $x_1, x_2 \in A$ be such that $f(x_1) = f(x_2)$
$\implies \dfrac{4x_1+3}{6x_1-4} = \dfrac{4x_2+3}{6x_2-4}$
$\implies 24x_1x_2 + 18x_2 - 16x_1 - 12 = 24x_1x_2 + 18x_1 - 16x_2 - 12$
$\implies 18x_2 - 16x_1 = 18x_1 - 16x_2$
$\implies 34x_2 = 34x_1 \implies x_1 = x_2 \implies f$ is one-one.
Let $y = f(x) \implies y = \dfrac{4x+3}{6x-4}$
$\implies 6xy - 4y = 4x + 3$
$\implies (6y-4)x = 4y + 3$
$\implies x = \dfrac{4y+3}{6y-4}$ which is defined for $y \neq \dfrac{2}{3}$
$\implies$ range of $f = \mathbf{R} - \left\{\dfrac{2}{3}\right\} = A$
$\implies$ range of f = co-domain of $f \implies f$ is onto.
Thus, the function f is one-one and onto.
$\implies f$ is invertible.
Now, $y = f(x)$
$\implies f^{-1}(y) = x = \dfrac{4y+3}{6y-4}$
Hence, $f^{-1}(x) = \dfrac{4x+3}{6x-4}$.

(b) Let $x_1, x_2 \in \mathbf{R}_+$ such that $f(x_1) = f(x_2)$

$\implies 5x_1^2 + 6x_1 - 9 = 5x_2 + 6x_2 - 9$

$\implies 5(x_1^2 - x_2^2) + 6(x_1 - x_2) = 0$

$\implies (x_1 - x_2)[5(x_1 + x_2) + 6] = 0$

$\implies x_1 - x_2 = 0 \quad [\because x_1, x_2 \in \mathbf{R}_+ \implies 5(x_1 + x_2) + 6 \neq 0]$

$\implies f$ is one-one.

As $x \in \mathbf{R}_+, x \geq 0$

$\implies 5x^2 + 6x \geq 0$

$\implies 5x^2 + 6x - 9 \geq -9$

$\implies$ range of $f = [-9, \infty)$

$\therefore$ Range of $f = [-9, \infty) =$ codomain of $f \implies f$ is onto.

Hence, the given function f is bijective.

$\therefore f$ is invertible.

Again, $f(x) = y$

$\implies 5x^2 + 6x - 9 = y$

$\implies 25x^2 + 30x - 45 = 5y$

$\implies (5x + 3)^2 - 9 - 45 = 5y$

$\implies (5x + 3)^2 = 54 + 5y$

$\implies 5x + 3 = \sqrt{5y + 54}$ (omitting the negative sign as, $x \geq 0$)

$\implies 5x = \sqrt{5y + 54} - 3$

$\implies x = \dfrac{\sqrt{5y + 54} - 3}{5}$

Now, $f(x) = y$ and f is invertible.

$\implies f^{-1}(y) = x = \dfrac{\sqrt{5y + 54} - 3}{5}$

$\implies f^{-1}(y) = \dfrac{\sqrt{5y + 54} - 3}{5}$.

Binary Operations

1. For each binary operation $'*'$ defined below, determine whether $'*'$ is binary, commutative and associative.

   (a) On $\mathbf{Q}$, defined by $a*b=\dfrac{ab}{2}$.

   (b) On $\mathbf{Q}$, defined by $a*b=a-b+ab$.

   (c) On $\mathbf{R}-\{-1\}$, defined by $a*b=\dfrac{a}{b+1}$.

Answers

(a) $a*b=\dfrac{ab}{2}$

Let $a,b\in\mathbf{Q}$ then $\dfrac{ab}{2}\in\mathbf{Q}$

Hence $'*'$ is binary.

Now, $a*b=\dfrac{ab}{2}=\dfrac{ba}{2}=b*a$

so, $'*'$ is commutative.

Now, $a*(b*c)=a*\left(\dfrac{bc}{2}\right)=\dfrac{abc}{2\times 2}=\dfrac{abc}{4}$

and, $(a*b)*c=\left(\dfrac{ab}{2}\right)*c=\dfrac{abc}{2\times 2}=\dfrac{abc}{4}$

Hence $a*(b*c)=(a*b)*c$

So, $'*'$ is associative.

(b) $a*b=a-b+ab, a,b\in\mathbf{Q}$

$a,b\in\mathbf{Q}\implies a-b+ab\in\mathbf{Q}$

Hence $'*'$ is binary.

Now, $a*b=a-b+ab\neq b-a+ab=b*a$

Hence, $'*'$ is not commutative.

Now, $a*(b*c)$

$=a*(b-c+bc)$

$=a-(b-c+bc)+a(b-c+bc)$

$=a-b+c-bc+ab-ac+abc$

And $(a*b)*c$

$=(a-b+ab)*c$

$a-b+ab-c+(a-b+ab)c$

$=a-b-c+ab+ac-bc+abc$

$\therefore a*(b*c)\neq(a*b)*c$

Hence $'*'$ is associative.

(c) $a*b=\dfrac{a}{b+1}$

Taking $a=1,b=-2,\quad \dfrac{a}{b+1}=-1\notin\mathbf{R}-\{-1\}$

Hence, $'*'$ is not binary.

Again, $a*b=\dfrac{a}{b+1}$ and $b*a=\dfrac{b}{a+1}$

But, $\dfrac{a}{b+1}\neq\dfrac{b}{a+1}$

$\implies a * b \neq b * a$
Hence $'*'$ is not commutative.
Now, $(a * b) * c = \left(\dfrac{a}{b+1}\right) * c$

$= \dfrac{\frac{a}{b+1}}{c+1}$

$= \dfrac{a}{(b+1)(c+1)}$

and, $a * (b * c) = a * \left(\dfrac{b}{c+1}\right)$

$= \dfrac{a}{\frac{b}{c+1}+1} = \dfrac{a}{\frac{b+c+1}{c+1}}$

$= \dfrac{a(c+1)}{b+c+1}$

$\therefore (a * b) * c \neq a * (b * c)$
Hence $*$ is not associative.

2. Let S be the set of all rational numbers except 1 and $'*'$ be defined on S by $a * b = a + b - ab \;\forall\; a, b \in S$. Prove that:

   (a) $'*'$ is a binary operation on S

   (b) The operation is commutative as well as associative.

   Find the identity element. Also find the inverse of an element $a \in A$.

   <u>**Answer**</u>

   (a) $a, b \in \mathbf{Q} - \{1\}$
   $\implies a + b - ab \in \mathbf{Q} - \{1\}$
   $\because$ If $a + b - ab = 1 \implies a(1-b) - (1-b) = 0 \implies (a-1)(1-b) = 0$ which is not possible as $a, b \in \mathbf{Q} - \{1\}$.
   Hence, $'*'$ is binary.

   (b) $a * b = a + b - ab$
   $= b + a - ba = b * a$
   Hence $'*'$ is commutative.
   $a * (b * c)$
   $= a * (b + c - bc)$
   $= a + b + c - bc + a(b + c - bc)$
   $= a + b + c - bc - ab - ac + abc$
   $(a * b) * c$
   $= (a + b - ab) * c$
   $= a + b - ab + c - (a + b - ab)c$
   $= a + b + c - ab - bc - ac + abc$
   $\therefore a * (b * c) = (a * b) * c$
   So, $'*'$ is associative.

   Let if possible $e \in S$ be such that $e * a = a \;\forall\; a \in S$.
   $\implies a + e - ae = a$

$\implies e(1-a)=0$
$\implies e=0\in S\ \ [\because a\neq 1]$
Hence, the identity element $e=0$ exists.
Let a^{-1} be the inverse of element a, then we must have, $a*a^{-1}=e$ (identity)
$\implies a+a^{-1}-a.a^{-1}=0$
$\implies a^{-1}(1-a)=-a$
$\implies a^{-1}=\dfrac{-a}{1-a}=\dfrac{a}{a-1}\in S\ \ \forall\ a\in S$
Hence, inverse of each element exists in S.

3. If $A=\mathbf{N}\times\mathbf{N}$ and $'*'$ on A is defined by $(a,b)*(c,d)=(ad+bc,bd)\ \ \forall\ (a,b),(c,d)\in A$, then show that:

 (a) $'*'$ is a binary operation on A
 (b) $'*'$ is commutative on A
 (c) $'*'$ is associative on A
 (d) A has no identity element with respect to the given operation.

Answer

 (a) Consider any $(a,b),(c,d)\in A \implies a,b,c,d\in\mathbf{N}$
 $\implies ad+bc,bd\in\mathbf{N} \implies (ad+bc,bd)\in A$
 $\implies (a,b)*(c,d)\in A$
 $\implies$ $'*'$ is a binary operation.

 (b) For all $(a,b),(c,d)\in A$ we have,
 $(a,b)*(c,d)=(ad+bc,bd)$, and
 $(c,d)*(a,b)=(cb+da,db)$
 $\because$ addition and multiplication are commutative on $\mathbf{N}$, $ad+bc=cb+da$ & $bd=db$.
 $\implies (ad+bc,bd)=(cb+da,db)$
 $\therefore (a,b)*(c,d)=(c,d)*(a,b)$
 $\implies$ The binary operation $'*'$ on A is commutative.

 (c) For all $(a,b),(c,d),(e,f)\in A$, we have
 $\{(a,b)*(c,d)\}*(e,f)$
 $=(ad+bc,bd)*(e,f)$
 $=((ad+bc)f+(bd)e,(bd)f)$
 $=(adf+bcf+bde,bdf)$
 Again, $(a,b)*\{(c,d)*(e,f)\}$
 $=(a,b)*(cf+de,df)$
 $=(a(df)+b(cf+de),b(df))$
 $=(adf+bcf+bde,bde)$
 $\implies \{(a,b)*(c,d)\}*(e,f)=(a,b)*\{(c,d)*(e,f)\}$
 $\implies$ the binary operation $'*'$ on A is associative.

 (d) Let if possible, $(e_1,e_2)\in A$ be the identity element in A, then we must have
 $(e_1,e_2)*(a,b)=(a,b)\ \forall\ (a,b)\in A$
 $\implies (e_1b+e_2a,e_2b)=(a,b)\ \forall\ (a,b)\in A$
 $be_1+e_2a=a$ and $be_2=b\ \forall\ a,b\in\mathbf{N}$
 $\implies e_1=0$ and $e_2=1$
 But $0\notin\mathbf{N}$ and therefore $(0,1)\notin A$
 Hence, A has no identity element with respect to the given binary operation.

4. Let $A = \mathbf{Q} \times \mathbf{Q}$, where $\mathbf{Q}$ is the set of rational numbers, and $*$ be a binary operation on A defined by $(a,b) * (c,d) = (ac, b+ad)\ \forall\ (a,b),(c,d) \in A$ then find:

   (a) the identity element of $'*'$ in A,

   (b) the invertible elements of A, and hence write the inverse of elements $(5,3)$ and $\left(\frac{1}{2}, 4\right)$.

Answer

(a) Let if possible, $(e_1, e_2) \in A$ be the identity element, then we must have,
$(a,b) * (e_1,e_2) = (a,b)\ \forall\ (a,b) \in A$
$\implies (ae_1, b+ae_2) = (a,b) \implies ae_1 = a$ and $b + ae_2 = b$
$\implies e_1 = 1$ and $e_2 = 0$, (provided $a \neq 0$)
$\therefore (e_1,e_2) = (1,0)$
If $a = 0$, then $(0,b) * (1,0) = (0 \times 1, b + (0 \times 0)) = (0,b)$
Also, $(1,0) * (0,b) = (1 \times 0, 0 + (1 \times b)) = (0,b)$
Thus, $(1,0)$ is such that:
$(1,0) * (a,b) = (a,b) = (a,b) * (1,0)\ \forall\ (a,b) \in A$
$\implies (1,0)$ is the identity element of given binary operation $'*'$ on A.

(b) Consider any element $(a,b) \in A$. If $(c,d) \in A$ be its inverse then we must have,
$(a,b) * (c,d) = (1,0)$
$\implies (ac, b+ad) = (1,0)$
$\implies ac = 1$ and $b + ad = 0$
$\implies c = \frac{1}{a}$ and $d = \frac{-b}{a}$, (provided $a \neq 0$)
Thus, (a,b) is invertible, provided $a \neq 0$
and the inverse of (a,b) is $\left(\frac{1}{a}, \frac{-b}{a}\right)$.
Hence, all elements $(a,b) \in A$ are invertible provided $a \neq 0$
$\therefore$ The inverse of $(5,3)$ is $\left(\frac{1}{5}, \frac{-3}{5}\right)$.
$\therefore$ The inverse of $\left(\frac{1}{2}, 4\right)$ is $\left(\frac{1}{\frac{1}{2}}, \frac{-4}{\frac{1}{2}}\right) = (2, -8)$.

Practise Problems

1. Show that the relation R in the set $A = \{1, 2, 3, 4, 5\}$ given by $R = \{(a, b) : |a - b| \text{ is even }\}$ is an equivalence relation.

2. Show that the function $f : \mathbf{R}^* \to \mathbf{R}^*$ defined by $f(x) = \frac{1}{x}$ is one-one and onto, where $\mathbf{R}^*$ is the set of all non-zero real numbers.

3. Show that the function $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = |x|$, is neither one-one nor onto.

4. Show that the signum function $f : \mathbf{R} \to \mathbf{R}$, given by $f(x) = \begin{cases} 1 & ,\text{if } x > 0 \\ 0 & ,\text{if } x = 0 \\ -1 & ,\text{if } x < 1 \end{cases}$ is neither one-one nor onto.

5. Find gof and fog, if $f(x) = 8x^3$ and $g(x) = x^{\frac{1}{3}}$. Ans: $2x, 8x$

6. Consider $f : \mathbf{R}^+ \to [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible and find the inverse, where $\mathbf{R}^+$ is the set of all non-negative real numbers.
   Ans: $f^{-1}(x) = \sqrt{x - 4}$

7. If $*$ is binary on $\mathbf{Q}$, defined by $a * b = \frac{3ab}{5}$. Show that $*$ is commutative as well as associative. Also, find its identity element if it exists. Ans: $e = \frac{5}{3}$

8. Is $*$ a binary operation on the set $\mathbf{Q}$ such that $a * b = (2a - b)^2$ for all $a, b \in \mathbf{Q}$?

9. If $f : \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = x^2 - 3x + 2$, find $f(f(x))$.
   Ans: $x^4 - 6x^3 + 10x^2 - 3x$

10. Let $f : \mathbf{N} \to \mathbf{Y}$ be a function defined as $f(x) = 4x + 3$, where, $Y = \{y \in \mathbf{N} : y = 4x + 3 \text{ for some } x \in \mathbf{N}\}$. Show that f is invertible. Find the inverse.
    Ans: $f^{-1}(x) = \frac{x - 3}{4}$